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A SUFFICIENT CONDITION FOR THE MAXIMUM NUMBER OF IMAGINARY ROOTS OF AN EQUATION OF THE N-TH DEGREE

BY E. B. VAN VLECK

So far as I have been able to ascertain, the following theorem is new:

If c_i is real and the terms of the sequence

$$c_0, \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}, \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}, \cdots\cdots, \Delta_n = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots\cdots & c_n \\ c_1 & c_2 & c_3 & \cdots\cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n+2} \end{vmatrix}, \tag{1}$$

are positive, all of the roots of the equation

$$c_0 + c_1x + c_2x^2 + \cdots\cdots + c_{2n}x^{2n} = 0 \tag{2}$$

are imaginary, and all but one of the roots of the equation

$$c_0 + c_1x + c_2x^2 + \cdots\cdots + c_{2n+1}x^{2n+1} = 0.^* \tag{3}$$

An extremely simple proof of this result can be obtained with the aid of a theorem of Sylvester† concerning the minors of Δ_n which I will state in the following restricted form: when Δ_n and the minors appearing in (1) are all positive, any minor formed symmetrically from Δ_n by erasing corresponding rows and columns will also be positive. In particular,

$$c_{2i} > 0, \qquad \begin{vmatrix} c_{2i} & c_{2i+1} \\ c_{2i+1} & c_{2i+2} \end{vmatrix} > 0. \tag{4}$$

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* This is somewhat like the well known theorem given by Borchardt in *Liouville's Journal*, vol. 12 (1846), p. 58. The latter uses a similar sequence of determinants constructed with s_i (the sum of the i-th powers of the roots) in place of c_i and continuing for nearly twice as many terms until the discriminant is reached. The number of pairs of imaginary roots is equal to the number of changes of sign in the sequence. Thus $2n$ roots will be imaginary if, and only if, there are n changes of sign. This test for the maximum number of imaginary roots has the advantage of being a necessary one, but, on the other hand, it is much less convenient in its application since it not only involves a greater number of terms but also necessitates the preliminary computation of the s_i.

†*Philosophical Transactions*, vol. 143 (1853), p. 482, foot-note. See also *Philosophical Magazine*, Ser. 4, vol. 4 (1852), p. 140-141.

Now we have identically

$$
\begin{aligned}
c_0 + c_1x + c_2x^2 + \cdots + c_{2n}x^{2n} = {} & c_0\left(1 + \frac{c_1}{2c_0}x\right)^2 + \frac{c_0c_2 - c_1^2}{4c_0}x^2 \\
& + \frac{3c_2x^2}{4}\left(1 + \frac{2}{3}\frac{c_3}{c_2}x\right)^2 + \frac{2}{6}\,\frac{c_2c_4 - c_3^2}{c_2}x^4 \\
& + \frac{4c_4x^4}{6}\left(1 + \frac{3}{4}\frac{c_5}{c_4}x\right)^2 + \frac{3}{8}\,\frac{c_4c_6 - c_5^2}{c_4}x^6 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
+ \frac{(n+1)c_{2n-2}x^{2n-2}}{2n}\left(1 + \frac{n}{n+1}\frac{c_{2n-1}}{c_{2n-2}}x\right)^2 & + \frac{n}{2(n+1)}\,\frac{c_{2n-2}c_{2n} - c_{2n-1}^2}{c_{2n-2}}x^{2n} \\
+ \frac{n+2}{2(n+1)}c_{2n}x^{2n}. &
\end{aligned}
$$

Since the right hand member consists of the sums of squares multiplied into positive coefficients, the roots of (2) must be imaginary.

To prove the second part of the theorem, put $x = 1/z$ and write equation (3) in the form

$$c_0z^{2n+1} + c_1z^{2n} + \cdots + c_{2n}z + c_{2n+1} = 0. \tag{5}$$

The derivative of the left hand member can be expressed as

$$
\begin{aligned}
& c_{2n}\left(1 + \frac{c_{2n-1}}{c_{2n}}z\right)^2 + \frac{c_{2n-2}c_{2n} - c_{2n-1}^2}{c_{2n}}z^2 \\
& + 2c_{2n-2}z^2\left(1 + \frac{c_{2n-3}}{c_{2n-2}}z\right)^2 + 2\,\frac{c_{2n-4}c_{2n-2} - c_{2n-3}^2}{c_{2n-2}}z^4 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + n\,c_2z^{2n-2}\left(1 + \frac{c_1}{c_2}z\right)^2 + n\,\frac{c_2c_0 - c_1^2}{c_2}z^{2n} + (n+1)c_0z^{2n}.
\end{aligned}
$$

But again, by (4), this is the sum of positive terms. Since the derivative of the left hand member of (5) is positive, it must be an increasing function of z and so vanishes but once when z traverses the real axis.

WESLEYAN UNIVERSITY, MAY, 1903.